# Trefftz-DG Approximation for the Elasto-Acoustics

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Depth Imaging Partnership

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# TREFFTZ-DG APPROXIMATION FOR THE ELASTO-ACOUSTICS

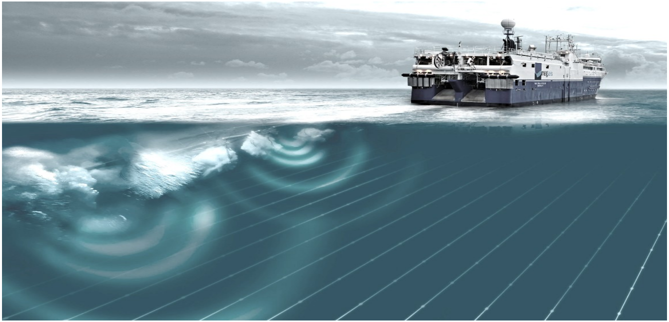
WAVES17 | 13th International Conference on Mathematical and Numerical Aspects of Wave Propagation

Minneapolis | 2017

# ABSTRACT

# SEISMIC SURVEY

FIGURE 1: Ocean Bottom Seismic (OBS) data acquisition



# BASIC NUMERICAL METHODS

TABLE 1: Generic properties of the most widely used numerical methods*

| Numerical method | Complex geometries | High-order accuracy and *hp*-adaptivity | Explicit semi-discrete form | Conservation laws | Elliptic problems |
|---|---|---|---|---|---|
| FDM | ● | ● | ● | ● | ● |
| FVM | | | | | |
| FEM | | | | | |
| DG-FEM | | | | | |

* J.S.Hesthaven T.Warburton. *Nodal DG methods. Algorithms, analysis, and applications.* 2007

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| FEM | ● | ● | ● | ⊙ | ● |
| DG-FEM | | | | | |

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| DG-FEM | ● | ● | ● | ● | ⊙ |

* J.S.Hesthaven T.Warburton. *Nodal DG methods. Algorithms, analysis, and applications.* 2007

# DISCONTINUOUS GALERKIN METHODS

## PROS AND CONS

Adapted to the complex geometries

High-order accuracy and hp-adaptivity

Explicit semi-dicrete form

Conservation laws

# DISCONTINUOUS GALERKIN METHODS

## PROS AND CONS

Adapted to the complex geometries

High-order accuracy and hp-adaptivity

Explicit semi-dicrete form

Conservation laws

Higher number of degrees of freedom

comparing to the methods with continuous approximation

# TREFFTZ METHOD

## DEFINITION *

Given a region of an Euclidean space of some partitions of that region,

a **Trefftz method** is any procedure for solving boundary value problems of PDE or systems of PDE, on such region,

**using solutions of that PDE or its adjoint**, defined in its subregions.

* I.Herrera. *Trefftz method: a general theory.* 2000

# DISCONTINUOUS GALERKIN METHODS

## TIME-HARMONIC FORMULATIONS

O.Cassenat B.Despres. 1998

T.Huttunen P.Monk J.P.Kaipo. 2002

C.Farhat I.Harari U.Hetmaniuk. 2003

R.Tezaur, C.Farhat. 2006

G.Gabard. 2007

## TIME-DOMAIN FORMULATIONS

H.Egger F.Kretzschmar S.M.Schnepp T.Weiland. 2014

F.Kretzschmar S.M.Schnepp I.Tsukerman T.Weiland. 2014

F.Kretzschmar A.Moiola I.Perugia. 2015

# TREFFTZ METHOD

## EXPECTED ADVANTAGES *

Better order of convergence

Flexibility in the choice of basis functions

Low dispersion

Adaptivity and local space-time mesh refinement

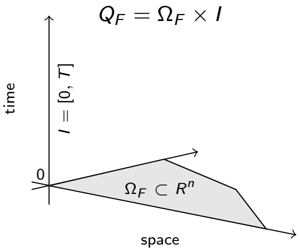
* I.Herrera. *Trefftz method: a general theory.* 2000

# MATHEMATICAL FORMULATION

## ACOUSTIC SYSTEM

# ACOUSTIC SYSTEM

## PROBLEM EQUATIONS

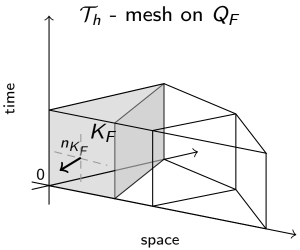



(AS)

$$\begin{cases} \dfrac{1}{c_F^2 \rho_F} \dfrac{\partial p}{\partial t} + \mathrm{div}\, v_F = f, & \text{in } Q_F \\ \rho_F \dfrac{\partial v_F}{\partial t} + \nabla p = 0, & \text{in } Q_F \\ v_F(\cdot, 0) = v_{F0}, \ \ p(\cdot, 0) = p_0, & \text{on } \Omega_F \\ v_F = g_F^D, & \text{on } \partial\Omega_F \times I \end{cases}$$

# ACOUSTIC SYSTEM

## MESHING




$\mathcal{F}_h = \cup_{K_F \in \mathcal{T}_h} \partial K_F$ - mesh skeleton

$K_F \in Q_F$ $(c_F, \rho_F \equiv const$ in $K_F)$

$v_F,\ p \in H^1(K_F)$

$\omega_F,\ q \in H^1(K_F)$

# ACOUSTIC SYSTEM

## TREFFTZ-DG FORMULATION IN THE ELEMENT $K_F$

$$\boxed{\begin{array}{c}\textbf{AS}\\(v_F, p)\end{array}}$$

# ACOUSTIC SYSTEM

## TREFFTZ-DG FORMULATION IN THE ELEMENT $K_F$

| AS $(v_F, p)$ | $\times$ | test functions $(\omega_F, q)$ |
|---|---|---|

# ACOUSTIC SYSTEM

## TREFFTZ-DG FORMULATION IN THE ELEMENT $K_F$

$$\underbrace{\boxed{\begin{matrix}\text{AS}\\(v_F,\, p)\end{matrix}} \times \boxed{\begin{matrix}\text{test functions}\\(\omega_F,\, q)\end{matrix}}}_{\text{SPACE-TIME INTEGRATION ON } K_F} =$$

# ACOUSTIC SYSTEM

## TREFFTZ-DG FORMULATION IN THE ELEMENT $K_F$

VOLUME INTEGRATION TERM

AS $\times$ $(v_F, p)$
(test functions)

AS $(v_F, p)$ $\times$ test functions $(\omega_F, q)$ $=$

SPACE-TIME INTEGRATION ON $K_F$

# ACOUSTIC SYSTEM

## TREFFTZ-DG FORMULATION IN THE ELEMENT $K_F$

| AS $(v_F, p)$ | $\times$ | test functions $(\omega_F, q)$ | $=$ | VOLUME INTEGRATION TERM: AS (test functions) $\times$ $(v_F, p)$ $+$ SURFACE INTEGRATION TERM: (test functions) $(v_F, p)$ |
|---|---|---|---|---|

SPACE-TIME INTEGRATION ON $K_F$

# ACOUSTIC SYSTEM

## TREFFTZ-DG FORMULATION IN THE ELEMENT $K_F$

VOLUME INTEGRATION TERM

AS $\times$ $(v_F, p)$
(test functions)

AS $(v_F, p)$ $\times$ test functions $(\omega_F, q)$ $=$

$+$

SPACE-TIME INTEGRATION ON $K_F$

SURFACE INTEGRATION TERM

(test functions) $(v_F, p)$

# ACOUSTIC SYSTEM

## TREFFTZ-DG FORMULATION IN THE ELEMENT $K_F$

VOLUME INTEGRATION TERM

test functions in $\mathbf{T}$

AS $(v_F, p)$ $\times$ test functions $(\omega_F, q)$ $=$

$+$

SURFACE INTEGRATION TERM

(test functions) $(v_F, p)$

SPACE-TIME INTEGRATION ON $K_F$

# ACOUSTIC SYSTEM

## TREFFTZ-DG FORMULATION IN THE ELEMENT $K_F$

VOLUME INTEGRATION TERM

$= 0$

$$\underbrace{\boxed{\begin{matrix}\text{AS}\\(v_F,\ p)\end{matrix}} \times \boxed{\begin{matrix}\text{test functions}\\(\omega_F,\ q)\end{matrix}}}_{\text{SPACE-TIME INTEGRATION ON } K_F} = \boxed{= 0} + \boxed{\begin{matrix}\text{(test functions)}\\(v_F, p)\end{matrix}}$$

SURFACE INTEGRATION TERM

# ACOUSTIC SYSTEM

## TREFFTZ-DG FORMULATION IN THE ELEMENT $K_F$

SURFACE INTEGRATION TERM

$$\underbrace{\boxed{\begin{array}{c}\text{AS}\\ (v_F\,,\ p)\end{array}} \times \boxed{\begin{array}{c}\text{test}\\ \text{functions}\\ (\omega_F\,,\ q)\end{array}}}_{\text{SPACE-TIME INTEGRATION ON } K_F} = \boxed{\begin{array}{c}\text{(test functions)}\\ (v_F\,,\, p)\end{array}}$$

# ACOUSTIC SYSTEM

## TREFFTZ-DG FORMULATION

$$\overbrace{\underbrace{\boxed{\text{AS}\ (v_F\,,\,p)} \times \boxed{\text{test functions}\ (\omega_F\,,\,q)}}_{\text{SPACE-TIME INTEGRATION ON } K_F}}^{\text{SUM OVER ALL } K_F} = \overbrace{\underset{\text{SURFACE INTEGRATION TERM}}{\boxed{(\text{test functions})\ (v_F\,,\,p)}}}^{\text{SUM OVER ALL } \partial K_F}$$

# ACOUSTIC SYSTEM

## SPACE-TIME DISCRETIZATION

$$-\sum_{K_F}\int_{K_F}\Big[p_h\big(\frac{1}{c_F^2\rho_F}\frac{\partial q}{\partial t}+\mathrm{div}\omega_F\big)+v_{F\,h}\cdot\big(\rho_F\frac{\partial\omega_F}{\partial t}+\nabla q\big)\Big]dv$$

$$+\sum_{K_F}\int_{\partial K_F}\Big[\big(\frac{1}{c_F^2\rho_F}\hat{p}_h q+\rho_F\hat{v_F}_h\cdot\omega_F\big)n^t_{K_F}+\big(\hat{p}_h\omega_F+\hat{v_F}_h q\big)\cdot n^x_{K_F}\Big]ds=\sum_{K_F}\int_{K_F}fq\,dv$$

# ACOUSTIC SYSTEM

## SPACE-TIME DISCRETIZATION

$$-\sum_{K_F}\int_{K_F}\Big[p_h\big(\frac{1}{c_F^2\rho_F}\frac{\partial q}{\partial t}+\mathrm{div}\omega_F\big)+v_{F\,h}\cdot\big(\rho_F\frac{\partial \omega_F}{\partial t}+\nabla q\big)\Big]d\mathrm{v}$$

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# ACOUSTIC SYSTEM

## SPACE-TIME DISCRETIZATION

$$-\sum_{K_F}\int_{K_F}\left[p_h\Big(\frac{1}{c_F^2\rho_F}\frac{\partial q}{\partial t}+\mathrm{div}\omega_F\Big)+v_{F\,h}\Big(\rho_F\frac{\partial\omega_F}{\partial t}+\nabla q\Big)\right]dv$$

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## TREFFTZ SPACE

$$\mathbf{T}_F(\mathcal{T}_h):=\left\{(\omega_F,q)\subset H^1(\mathcal{T}_h)^2:\forall K_F\in\mathcal{T}_h,\ \rho_F\frac{\partial\omega_F}{\partial t}+\nabla q=\frac{1}{c_F^2\rho_F}\frac{\partial q}{\partial t}+\mathrm{div}\omega_F=0\right\}$$

# ACOUSTIC SYSTEM

## SPACE-TIME DISCRETIZATION

$$-\sum_{K_F}\int_{K_F}\Big[p_h\Big(\frac{1}{c_F^2\rho_F}\frac{\partial q}{\partial t}+\mathrm{div}\omega_F\Big)+v_{F\,h}\Big(\rho_F\frac{\partial \omega_F}{\partial t}+\nabla q\Big)\Big]d\mathrm{v}$$

$$+\sum_{K_F}\int_{\partial K_F}\Big[\Big(\frac{1}{c_F^2\rho_F}\hat{p}_h q+\rho_F\hat{v_F}_h\omega_F\Big)n^t_{K_F}+(\hat{p}_h\omega_F+\hat{v_F}_h q)n^x_{K_F}\Big]d\mathrm{s}=\sum_{K_F}\int_{K_F} fq d\mathrm{v}$$

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## SPACE-TIME DISCRETIZATION IN $\mathrm{T}_F$

$$\sum_{K_F}\int_{\partial K_F}\Big[\Big(\frac{1}{c_F^2\rho_F}\hat{p}_h q+\rho_F\hat{v_F}_h\cdot\omega_F\Big)n^t_{K_F}+(\hat{p}_h\omega_F+\hat{v_F}_h q)\cdot n^x_{K_F}\Big]d\mathrm{s}=\sum_{K_F}\int_{K_F} fq d\mathrm{v}$$

# ACOUSTIC SYSTEM

## TREFFTZ-DG FORMULATION

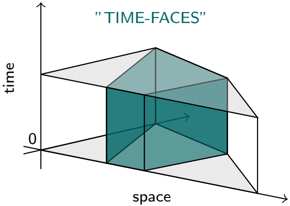
SUM OVER
ALL $\partial K_F$

$\boxed{\text{SURFACE INTEGRATION TERM}}$

# ACOUSTIC SYSTEM

## TREFFTZ-DG FORMULATION

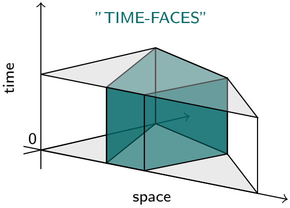
$$\underbrace{\boxed{\text{SURFACE INTEGRATION TERM}}}_{\text{SUM THROUGH ALL } \partial K_F} = \underbrace{\boxed{\text{SURFACE INTEGRATION TERM}}}_{\text{SUM THROUGH ALL "TIME-FACES"}}$$

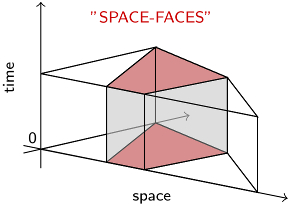


# ACOUSTIC SYSTEM

## TREFFTZ-DG FORMULATION

$$\underbrace{\boxed{\text{SURFACE INTEGRATION TERM}}}_{\text{SUM THROUGH ALL } \partial K_F} = \underbrace{\boxed{\text{SURFACE INTEGRATION TERM}}}_{\text{SUM THROUGH ALL "TIME-FACES"}} + \underbrace{\boxed{\text{SURFACE INTEGRATION TERM}}}_{\text{SUM THROUGH ALL "SPACE-FACES"}}$$

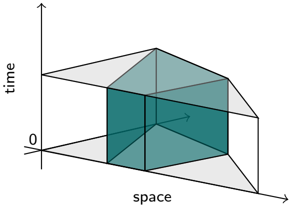


# ACOUSTIC SYSTEM

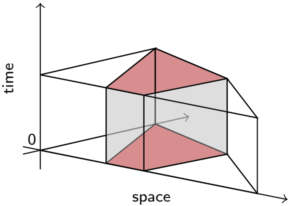
## MESH NOTATIONS

"TIME-FACES"




| | |
|---|---|
| internal | $\mathcal{F}_h^{I_F}$ |
| boundary | $\mathcal{F}_h^{D_F}$ $(\partial\Omega_F \times [0, T])$ |

"SPACE-FACES"




| | |
|---|---|
| internal | $\mathcal{F}_h^{\Omega_F}$ $(t \equiv const.)$ |
| initial | $\mathcal{F}_h^{0_F}$ $(\Omega_F \times \{0\})$ |
| final | $\mathcal{F}_h^{T_F}$ $(\Omega_F \times \{T\})$ |

# ACOUSTIC SYSTEM

## TREFFTZ-DG FORMULATION

$$\sum_{\partial K_F} \int_{\partial K_F} \Big[ \big( \frac{1}{c_F^2 \rho_F} \hat{p}_h q + \rho_F \hat{v_F}_h \omega_F \big) n_{K_F}^t + (\hat{p}_h \omega_F + \hat{v_F}_h q) n_{K_F}^x \Big] ds = \sum_{K_F} \int_{K_F} f q d\mathrm{v}$$

## NUMERICAL FLUXES THROUGH THE ELEMENT FACES

$$\mathcal{F}_h^{I_F}: \quad \begin{pmatrix} \hat{v_F}_h \\ \hat{p}_h \end{pmatrix} := \begin{pmatrix} \{v_{F\,h}\} + \beta [\![ p_h ]\!]_x \\ \{p_h\} + \alpha [\![ v_{F\,h} ]\!]_x \end{pmatrix}$$

$$\mathcal{F}_h^{D_F}: \quad \begin{pmatrix} \hat{v_F}_h \\ \hat{p}_h \end{pmatrix} := \begin{pmatrix} g_{\hat{D}_F} \\ p_h + \alpha (v_{F\,h} \cdot n_{K_F}^x - g_{D_F}) \end{pmatrix}$$

$$\mathcal{F}_h^{\Omega_F}: \quad \begin{pmatrix} \hat{v_F}_h \\ \hat{p}_h \end{pmatrix} := \begin{pmatrix} {v_{F\,h}}^- \\ {p_h}^- \end{pmatrix}$$

$$\mathcal{F}_h^{T_F}: \quad \begin{pmatrix} \hat{v_F}_h \\ \hat{p}_h \end{pmatrix} := \begin{pmatrix} v_{F\,h} \\ p_h \end{pmatrix}$$

$$\mathcal{F}_h^{0_F}: \quad \begin{pmatrix} \hat{v_F}_h \\ \hat{p}_h \end{pmatrix} := \begin{pmatrix} v_{F0} \\ p_0 \end{pmatrix}$$

# ACOUSTIC SYSTEM

## TREFFTZ-DG FORMULATION

Seek $(v_{F\,h}, p_h) \subset \mathbf{V}(\mathcal{T}_h) \subset \mathbf{T}_F(\mathcal{T}_h)$ s.t. for all $(\omega_F, q) \subset \mathbf{V}(\mathcal{T}_h)$ it holds:

$$\mathcal{A}_{TDG_F}((v_{F\,h}, p_h); (\omega_F, q)) = \ell_{TDG_F}(\omega_F, q)$$

# ACOUSTIC SYSTEM

## $L^2(\mathcal{T}_h)$ NORMS IN $\mathrm{T}_F$

$$
\begin{aligned}
|||(\omega_F, q)|||^2_{TDG_F} := & \frac{1}{2}\left\|\left(\frac{1}{c_F^2\rho_F}\right)^{1/2}[\![q]\!]_t\right\|^2_{L^2(\mathcal{F}_h^{\Omega_F})} + \frac{1}{2}\left\|\rho_F^{1/2}[\![\omega_F]\!]_t\right\|^2_{L^2(\mathcal{F}_h^{\Omega_F})} + \left\|\alpha^{1/2}[\![\omega_F]\!]_x\right\|^2_{L^2(\mathcal{F}_h^{I_F})} \\
& + \left\|\beta^{1/2}[\![q]\!]_x\right\|^2_{L^2(\mathcal{F}_h^{I_F})} + \frac{1}{2}\left\|\left(\frac{1}{c_F^2\rho_F}\right)^{1/2} q\right\|^2_{L^2(\mathcal{F}_h^{T_F})} + \frac{1}{2}\left\|\rho_F^{1/2}\omega_F\right\|^2_{L^2(\mathcal{F}_h^{T_F})} \\
& + \left\|\alpha^{1/2}\omega_F\right\|^2_{L^2(\mathcal{F}_h^{D_F})}
\end{aligned}
$$

# ACOUSTIC SYSTEM

## $L^2(\mathcal{T}_h)$ NORMS IN $\mathrm{T}_F$

$$
\begin{aligned}
|||(\omega_F, q)|||^2_{TDG_F} :=& \frac{1}{2}\left\|\left(\frac{1}{c_F^2\rho_F}\right)^{1/2}[\![q]\!]_t\right\|^2_{L^2(\mathcal{F}_h^{\Omega_F})} + \frac{1}{2}\left\|\rho_F^{1/2}[\![\omega_F]\!]_t\right\|^2_{L^2(\mathcal{F}_h^{\Omega_F})} + \left\|\alpha^{1/2}[\![\omega_F]\!]_x\right\|^2_{L^2(\mathcal{F}_h^{I_F})} \\
&+ \left\|\beta^{1/2}[\![q]\!]_x\right\|^2_{L^2(\mathcal{F}_h^{I_F})} + \frac{1}{2}\left\|\left(\frac{1}{c_F^2\rho_F}\right)^{1/2} q\right\|^2_{L^2(\mathcal{F}_h^{T_F})} + \frac{1}{2}\left\|\rho_F^{1/2}\omega_F\right\|^2_{L^2(\mathcal{F}_h^{T_F})} \\
&+ \left\|\alpha^{1/2}\omega_F\right\|^2_{L^2(\mathcal{F}_h^{D_F})}
\end{aligned}
$$

$$
\begin{aligned}
|||(\omega_F, q)|||^2_{TDG_F^*} :=& |||(\omega_F, q)|||^2_{TDG_F} \\
&+ \|\rho_F^{1/2}\omega_F^-\|^2_{L^2(\mathcal{F}_h^{\Omega_F})} + \|\left(\frac{1}{c_F^2\rho_F}\right)^{1/2} q^-\|^2_{L^2(\mathcal{F}_h^{\Omega_F})} + \|\beta^{-1/2}\{\omega_F\}\|^2_{L^2(\mathcal{F}_h^{I_F})} \\
&+ \|\alpha^{-1/2}\{q\}\|^2_{L^2(\mathcal{F}_h^{I_F})} + \|\alpha^{-1/2}\omega_F\|^2_{L^2(\mathcal{F}_h^{D_F})}
\end{aligned}
$$

# ACOUSTIC SYSTEM

## COERCIVITY OF TREFFTZ-DG FORMULATION

$$\mathcal{A}_{TDG_F}((\omega_F, q); (\omega_F, q)) = |||(\omega_F, q)|||^2_{TDG_F}, \ \forall (\omega_F, q) \in \mathbf{T}_F(\mathcal{T}_h)$$

# ACOUSTIC SYSTEM

## COERCIVITY OF TREFFTZ-DG FORMULATION

$$\mathcal{A}_{TDG_F}((\omega_F, q); (\omega_F, q)) = |||(\omega_F, q)|||^2_{TDG_F}, \quad \forall (\omega_F, q) \in \mathbf{T}_F(\mathcal{T}_h)$$

## CONTINUITY OF TREFFTZ-DG FORMULATION

$$|\mathcal{A}_{TDG_F}((v_F, p); (\omega_F, q))| \leq 2\, |||(v_F, p)|||_{TDG_F^*}\, |||(\omega_F, q)|||_{TDG_F},$$

$$|\ell_{TDG_F}(\omega_F, q)| \leq \sqrt{2} \left[ \|\rho_F^{1/2} v_{f0}\|^2_{L^2(\mathcal{F}_h^{0_F})} + \|(\frac{1}{c_F^2 \rho_F})^{1/2} p_0\|^2_{L^2(\mathcal{F}_h^{0_F})} \right]^{1/2}$$

# ACOUSTIC SYSTEM

## WELL-POSEDNESS

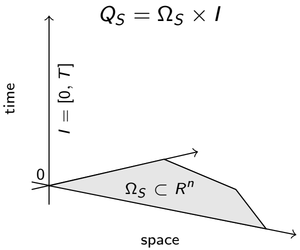
$$|||(v_F - v_{F\,h}, p - p_h)|||_{TDG_F} \leq 3 \inf_{(\omega_F, q) \in \mathbf{V}(\mathcal{T}_h)} |||(v_F - \omega_F, p - q)|||_{TDG_F^*}$$

# MATHEMATICAL FORMULATION

## ELASTODYNAMIC SYSTEM

# ELASTODYNAMIC SYSTEM

## PROBLEM EQUATIONS

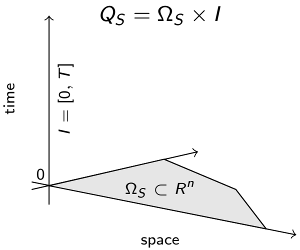



**(ES)**

$$\left[\begin{array}{ll} \dfrac{\partial \underline{\underline{\sigma}}}{\partial t} - \underline{\underline{C}}\,\underline{\underline{\varepsilon}}(v_S) = 0 & \text{in } Q_S \\ \rho_S \dfrac{\partial v_S}{\partial t} - \mathrm{div}\underline{\underline{\sigma}} = 0 & \text{in } Q_S \\ & \\ v_S(\cdot, 0) = v_{S0}, \ \underline{\underline{\sigma}}(\cdot, 0) = \underline{\underline{\sigma}}_0 & \text{on } \Omega_S \\ v_S = g_{D_S} & \text{on } \partial\Omega_S \times I \end{array}\right.$$

# ELASTODYNAMIC SYSTEM

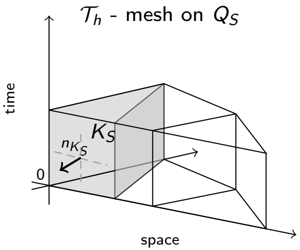
## PROBLEM EQUATIONS




$$
(ES) \quad \left[ \begin{array}{ll}
\underline{\underline{A}} \dfrac{\partial \underline{\underline{\sigma}}}{\partial t} - \underline{\underline{\varepsilon}}(v_S) = 0 & \text{in } Q_S \\
\rho_S \dfrac{\partial v_S}{\partial t} - \mathrm{div} \underline{\underline{\sigma}} = 0 & \text{in } Q_S \\
 & \\
v_S(\cdot, 0) = v_{S0}, \; \underline{\underline{\sigma}}(\cdot, 0) = \underline{\underline{\sigma}}_0 & \text{on } \Omega_S \\
v_S = g_{D_S} & \text{on } \partial\Omega_S \times I
\end{array} \right.
$$

# ELASTODYNAMIC SYSTEM

## MESHING




$\mathcal{F}_h = \cup_{K_S \in \mathcal{T}_h} \partial K_S$ - mesh skeleton

$K_S \in Q_S$ $(A, \rho_S \equiv const$ in $K_S)$

$v_S,\ \underline{\underline{\sigma}} \in H^1(K_S)$

$\omega_S,\ \underline{\underline{\xi}} \in H^1(K_S)$

# ELASTODYNAMIC SYSTEM

## TREFFTZ SPACE

$$\mathbf{T}_S(\mathcal{T}_h) := \left\{(\omega_S, \underline{\underline{\xi}}) \subset H^1(\mathcal{T}_h)^2 : \forall K_S \in \mathcal{T}_h,\ \rho_S \frac{\partial \omega_S}{\partial t} - \mathrm{div}\underline{\underline{\xi}} = \underline{\underline{A}} \frac{\partial \underline{\underline{\xi}}}{\partial t} - \underline{\underline{\varepsilon}}(\omega_S) = 0\right\}$$

# ELASTODYNAMIC SYSTEM

## TREFFTZ SPACE

$$\mathbf{T}_S(\mathcal{T}_h) := \left\{(\omega_S, \underline{\underline{\xi}}) \subset H^1(\mathcal{T}_h)^2 : \forall K_S \in \mathcal{T}_h,\ \rho_S \frac{\partial \omega_S}{\partial t} - \mathrm{div}\underline{\underline{\xi}} = \underline{\underline{A}} \frac{\partial \underline{\underline{\xi}}}{\partial t} - \underline{\underline{\varepsilon}}(\omega_S) = 0 \right\}$$

## SPACE-TIME DISCRETIZATION IN $\mathrm{T}_S$

$$\sum_{K_S} \int_{\partial K_S} \left[ (\underline{\underline{A}}\,\hat{\underline{\underline{\sigma}}}_h : \underline{\underline{\xi}} + \rho_S v \hat{s}_h \omega_S) n^t_{K_S} - (v \hat{s}_h \underline{\underline{\xi}} + \hat{\underline{\underline{\sigma}}}_h \omega_S) \cdot n^x_{K_S} \right] d\mathrm{s} = 0$$

# ELASTODYNAMIC SYSTEM

## TREFFTZ-DG FORMULATION

$$\sum_{K_S} \int_{\partial K_S} \left[ (\underline{\underline{A}}\,\hat{\underline{\underline{\sigma}}}_h : \underline{\underline{\xi}} + \rho_S \hat{v_S}_h \omega_S) n^t_{K_S} - (\hat{v_S}_h \underline{\underline{\xi}} + \hat{\underline{\underline{\sigma}}}_h \omega_S) \cdot n^x_{K_S} \right] ds = 0$$

## NUMERICAL FLUXES THROUGH THE ELEMENT FACES

$$\mathcal{F}_h^{I_S}: \quad \begin{pmatrix} \hat{v_S}_h \\ \hat{\underline{\underline{\sigma}}}_h \end{pmatrix} := \begin{pmatrix} \{v_{S\,h}\} - \delta[\![\underline{\underline{\sigma}}_h]\!]_x \\ \{\underline{\underline{\sigma}}_h\} - \gamma[\![v_{S\,h}]\!]_x \end{pmatrix}$$

$$\mathcal{F}_h^{D_S}: \quad \begin{pmatrix} \hat{v_S}_h \\ \hat{\underline{\underline{\sigma}}}_h \end{pmatrix} := \begin{pmatrix} v_{S\,h} - \delta(\underline{\underline{\sigma}}_h \cdot n^x_{K_S} - g_{D_S}) \\ g_{D_S} \end{pmatrix}$$

$$\mathcal{F}_h^{\Omega_S}: \quad \begin{pmatrix} \hat{v_S}_h \\ \hat{\underline{\underline{\sigma}}}_h \end{pmatrix} := \begin{pmatrix} {v_{S\,h}}^- \\ {\underline{\underline{\sigma}}_h}^- \end{pmatrix}$$

$$\mathcal{F}_h^{T_S}: \quad \begin{pmatrix} \hat{v_S}_h \\ \hat{\underline{\underline{\sigma}}}_h \end{pmatrix} := \begin{pmatrix} v_{S\,h} \\ \underline{\underline{\sigma}}_h \end{pmatrix}$$

$$\mathcal{F}_h^{0_S}: \quad \begin{pmatrix} \hat{v_S}_h \\ \hat{\underline{\underline{\sigma}}}_h \end{pmatrix} := \begin{pmatrix} v_{S\,0} \\ \underline{\underline{\sigma}}_0 \end{pmatrix}$$

# ELASTODYNAMIC SYSTEM

## TREFFTZ-DG FORMULATION

Seek $(v_{S\,h}, \underline{\underline{\sigma}}_h) \subset \mathbf{V}(\mathcal{T}_h) \subset \mathbf{T}_S(\mathcal{T}_h)$, s.t. for all $K_S \in \mathcal{T}_h$, $(\omega_S, \underline{\underline{\xi}}) \subset \mathbf{V}(\mathcal{T}_h)$ it holds:

$$\mathcal{A}_{TDG_S}((v_{S\,h}, \underline{\underline{\sigma}}_h); (\omega_S, \underline{\underline{\xi}})) = \ell_{TDG_S}(\omega_S, \underline{\underline{\xi}})$$

# ELASTODYNAMIC SYSTEM

## $L^2(\mathcal{T}_h)$ NORMS IN $T_S$

$$
\begin{aligned}
|||(\omega_S, \underline{\underline{\xi}})|||^2_{TDG_S} := & \frac{1}{2}\left\|(\underline{\underline{A}})^{1/2}[\![\underline{\underline{\xi}}]\!]_t\right\|^2_{L^2(\mathcal{F}_h^{\Omega_S})} + \frac{1}{2}\left\|\rho_S^{1/2}[\![\omega_S]\!]_t\right\|^2_{L^2(\mathcal{F}_h^{\Omega_S})} + \left\|\gamma^{1/2}[\![\omega_S]\!]_x\right\|^2_{L^2(\mathcal{F}_h^{I_S})} \\
& + \left\|\delta^{1/2}[\![\underline{\underline{\xi}}]\!]_x\right\|^2_{L^2(\mathcal{F}_h^{I_S})} + \frac{1}{2}\left\|(\underline{\underline{A}})^{1/2}\underline{\underline{\xi}}\right\|^2_{L^2(\mathcal{F}_h^{T_S})} + \frac{1}{2}\left\|\rho_S^{1/2}\omega_S\right\|^2_{L^2(\mathcal{F}_h^{T_S})} \\
& + \left\|\delta^{1/2}\underline{\underline{\xi}}\right\|^2_{L^2(\mathcal{F}_h^{D_S})}
\end{aligned}
$$

# ELASTODYNAMIC SYSTEM

## $L^2(\mathcal{T}_h)$ NORMS IN $\mathrm{T}_S$

$$
\begin{aligned}
|||(\omega_S,\underline{\underline{\xi}})|||^2_{TDG_S} := & \frac{1}{2}\left\|(\underline{\underline{A}})^{1/2}[\![\underline{\underline{\xi}}]\!]_t\right\|^2_{L^2(\mathcal{F}_h^{\Omega_S})} + \frac{1}{2}\left\|\rho_S^{1/2}[\![\omega_S]\!]_t\right\|^2_{L^2(\mathcal{F}_h^{\Omega_S})} + \left\|\gamma^{1/2}[\![\omega_S]\!]_x\right\|^2_{L^2(\mathcal{F}_h^{I_S})} \\
& + \left\|\delta^{1/2}[\![\underline{\underline{\xi}}]\!]_x\right\|^2_{L^2(\mathcal{F}_h^{I_S})} + \frac{1}{2}\left\|(\underline{\underline{A}})^{1/2}\underline{\underline{\xi}}\right\|^2_{L^2(\mathcal{F}_h^{T_S})} + \frac{1}{2}\left\|\rho_S^{1/2}\omega_S\right\|^2_{L^2(\mathcal{F}_h^{T_S})} \\
& + \left\|\delta^{1/2}\underline{\underline{\xi}}\right\|^2_{L^2(\mathcal{F}_h^{D_S})}
\end{aligned}
$$

$$
\begin{aligned}
|||(\omega_S,\underline{\underline{\xi}})|||^2_{TDG_S^*} := & |||(\omega_S,\underline{\underline{\xi}})|||^2_{TDG_S} \\
& + \|\rho_S^{1/2}\omega_S^-\|^2_{L^2(\mathcal{F}_h^{\Omega_S})} + \|(\underline{\underline{A}})^{1/2}\underline{\underline{\xi}}^-\|^2_{L^2(\mathcal{F}_h^{\Omega_S})} + \|\delta^{-1/2}\{\!\{\omega_S\}\!\}\|^2_{L^2(\mathcal{F}_h^{I_S})} \\
& + \|\gamma^{-1/2}\{\!\{\underline{\underline{\xi}}\}\!\}\|^2_{L^2(\mathcal{F}_h^{I_S})} + \|\delta^{-1/2}\underline{\underline{\xi}}\|^2_{L^2(\mathcal{F}_h^{D_S})}.
\end{aligned}
$$

# ELASTODYNAMIC SYSTEM

## COERCIVITY OF TREFFTZ-DG FORMULATION

$$\mathcal{A}_{TDG_S}((\omega_S, \underline{\underline{\xi}}); (\omega_S, \underline{\underline{\xi}})) = |||(\omega_S, \underline{\underline{\xi}})|||^2_{TDG_S}, \ \forall (\omega_S, \underline{\underline{\xi}}) \in \mathbf{T}_S(\mathcal{T}_h)$$

# ELASTODYNAMIC SYSTEM

## COERCIVITY OF TREFFTZ-DG FORMULATION

$$\mathcal{A}_{TDG_S}((\omega_S, \underline{\underline{\xi}}); (\omega_S, \underline{\underline{\xi}})) = |||(\omega_S, \underline{\underline{\xi}})|||^2_{TDG_S}, \quad \forall (\omega_S, \underline{\underline{\xi}}) \in \mathbf{T}_S(\mathcal{T}_h)$$

## CONTINUITY OF TREFFTZ-DG FORMULATION

$$|\mathcal{A}_{TDG_S}((v_S, \underline{\underline{\sigma}}); (\omega_S, \underline{\underline{\xi}}))| \leq 2\, |||(v_S, \underline{\underline{\sigma}})|||_{TDG_S^*}\, |||(\omega_S, \underline{\underline{\xi}})|||_{TDG_S}$$

$$|\ell_{TDG_S}(\omega_S, \underline{\underline{\xi}})| \leq \sqrt{2} \left[ \|\rho_S^{1/2} v_{s0}\|^2_{L^2(\mathcal{F}_h^{0S})} + \|\underline{\underline{A}}^{1/2} \underline{\underline{\sigma}}_0\|^2_{L^2(\mathcal{F}_h^{0S})} \right]^{1/2}$$

# ELASTODYNAMIC SYSTEM

## WELL-POSEDNESS

$$|||(v_S - v_{S\,h}, \underline{\underline{\sigma}} - \underline{\underline{\sigma}}_h)|||_{TDG_S} \leq 3 \inf_{(\omega_S, \underline{\underline{\xi}}) \in \mathbf{V}(\mathcal{T}_h)} |||(v_S - \omega_S, \underline{\underline{\sigma}} - \underline{\underline{\xi}})|||_{TDG_S^*}$$

# MATHEMATICAL FORMULATION

## ELASTO-ACOUSTIC SYSTEM

# ELASTO-ACOUSTIC SYSTEM

## NUMERICAL COUPLING ADVANTAGES

- Computing 1 unknown scalar pressure instead of 6 components of stress in fluid

# ELASTO-ACOUSTIC SYSTEM

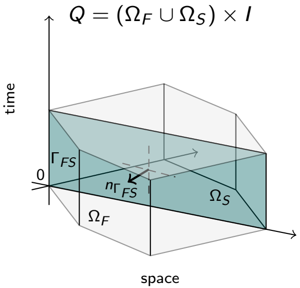
## NUMERICAL COUPLING ADVANTAGES

Computing 1 unknown scalar pressure instead of 6 components of stress in fluid

Avoiding the numerical artefacts caused by the slow $S$ - waves

# ELASTO-ACOUSTIC SYSTEM

## TRANSMISSION CONDITIONS




$$
(\mathrm{TC}) \quad \left[ \begin{array}{lll} v_F\, n_{\Gamma_{FS}} = v_S\, n_{\Gamma_{FS}} & & \text{on } \Gamma_{FS} \\ \underline{\underline{\sigma}} \cdot n_{\Gamma_{FS}} = -p n_{\Gamma_{FS}} & & \text{on } \Gamma_{FS} \end{array} \right.
$$

# ELASTO-ACOUSTIC SYSTEM

## TREFFTZ SPACE

$$\mathbf{T}(\mathcal{T}_h) := \Big\{(\omega_F, q, \omega_S, \underline{\underline{\xi}}) \subset H^1(\mathcal{T}_h)^4 : \ \rho_F \frac{\partial \omega_F}{\partial t} + \nabla q = \frac{1}{c_F^2 \rho_F} \frac{\partial q}{\partial t} + \mathrm{div}\omega_F = 0,$$

$$\rho_S \frac{\partial \omega_S}{\partial t} - \mathrm{div}\underline{\underline{\xi}} = \frac{\partial \underline{\underline{A}}\,\underline{\underline{\xi}}}{\partial t} - \underline{\underline{\varepsilon}}(\omega_S) = 0, \ \forall K_F, K_S \in \mathcal{T}_h\Big\}$$

# ELASTO-ACOUSTIC SYSTEM

## TREFFTZ SPACE

$$\mathbf{T}(\mathcal{T}_h) := \Big\{(\omega_F, q, \omega_S, \underline{\underline{\xi}}) \subset H^1(\mathcal{T}_h)^4 : \ \rho_F \frac{\partial \omega_F}{\partial t} + \nabla q = \frac{1}{c_F^2 \rho_F} \frac{\partial q}{\partial t} + \mathrm{div}\omega_F = 0,$$

$$\rho_S \frac{\partial \omega_S}{\partial t} - \mathrm{div}\underline{\underline{\xi}} = \frac{\partial \underline{\underline{A}}\,\underline{\underline{\xi}}}{\partial t} - \underline{\underline{\varepsilon}}(\omega_S) = 0, \ \forall K_F, K_S \in \mathcal{T}_h \Big\}$$

## SPACE-TIME DISCRETIZATION IN T

$$\sum_{K_F} \int_{\partial K_F} \Big[ \big( \frac{1}{c_F^2 \rho_F} \hat{p}_h q + \rho_F \hat{vF}_h \omega_F \big) n_{K_F}^t + (\hat{p}_h \omega_F + \hat{vF}_h q) n_{K_F}^x \Big] ds$$

$$+ \sum_{K_S} \int_{\partial K_S} \Big[ \big( \underline{\underline{A}}\,\hat{\underline{\underline{\sigma}}}_h : \underline{\underline{\xi}} + \rho_S \hat{vS}_h \omega_s \big) n_{K_S}^t - \big( \hat{vS}_h \underline{\underline{\xi}} + \hat{\underline{\underline{\sigma}}}_h \omega_S \big) \cdot n_{K_S}^x \Big] ds = \sum_{K_F} \int_{K_F} f q \, dv$$

# ELASTO-ACOUSTIC SYSTEM

## TREFFTZ-DG FORMULATION

$$\sum_{K_F} \int_{\partial K_F} \Big[ \big( \frac{1}{c_F^2 \rho_F} \hat{p}_h q + \rho_F \hat{v_F}_h \cdot \omega_F \big) n^t_{K_F} + \big( \hat{p}_h \omega_F + \hat{v_F}_h q \big) \cdot n^x_{K_F} \Big] ds,$$

$$+ \sum_{K_S} \int_{\partial K_S} \Big[ \big( \underline{\underline{A}}\, \hat{\underline{\underline{\sigma}}}_h : \underline{\underline{\xi}} + \rho_S \hat{v_S}_h \cdot \omega_s \big) n^t_{K_S} - \big( \underline{\underline{\xi}}\, \hat{v_S}_h + \hat{\underline{\underline{\sigma}}}_h \omega_S \big) \cdot n^x_{K_S} \Big] ds = \sum_{K_F} \int_{K_F} f q \, dv$$

## NUMERICAL FLUXES THROUGH THE FLUID-SOLID INTERFACE

$$\mathcal{F}_h^{FS}: \qquad \begin{pmatrix} \hat{v_F}_h n^x_{K_F} \\ \hat{p}_h \\ \hat{v_S}_h \\ \hat{\underline{\underline{\sigma}}}_h \cdot n^x_{K_S} \end{pmatrix} := \begin{pmatrix} {v_S}_h n^x_{K_F} \\ p_h + \alpha({v_F}_h n^x_{K_F} - {v_S}_h n^x_{K_F}) \\ {v_S}_h - \delta(\underline{\underline{\sigma}}_h \cdot n^x_{K_S} - p_h n^x_{K_S}) \\ -p_h n^x_{K_S} \end{pmatrix}$$

# ELASTO-ACOUSTIC SYSTEM

## TREFFTZ-DG FORMULATION

Seek $(v_{F\,h}, p_h, v_{S\,h}, \underline{\underline{\sigma}}_h) \subset \mathbf{V}(\mathcal{T}_h)$: $\forall K_F, K_S \in \mathcal{T}_h$, $(\omega_F, q, \omega_S, \underline{\underline{\xi}}) \subset \mathbf{T}(\mathcal{T}_h)$ it holds:

$$\mathcal{A}_{TDG}((v_{F\,h}, p_h, v_{S\,h}, \underline{\underline{\sigma}}_h); (\omega_F, q, \omega_S, \underline{\underline{\xi}})) = \ell_{TDG}(\omega_F, q, \omega_S, \underline{\underline{\xi}})$$

# ELASTO-ACOUSTIC SYSTEM

## $L^2(\mathcal{T}_h)$ NORMS IN T

$$|||(\omega_F, q, \omega_S, \underline{\underline{\xi}})|||^2_{TDG} := |||(\omega_F, q)|||^2_{TDG_F} + |||(\omega_S, \underline{\underline{\xi}})|||^2_{TDG_S} + 2\|\delta^{1/2}\underline{\underline{\xi}}\|^2_{L^2(\mathcal{F}_h^{FS})}$$

# ELASTO-ACOUSTIC SYSTEM

## $L^2(\mathcal{T}_h)$ NORMS IN T

$$|||(\omega_F, q, \omega_S, \underline{\underline{\xi}}))|||^2_{TDG} := |||(\omega_F, q)|||^2_{TDG_F} + |||(\omega_S, \underline{\underline{\xi}})|||^2_{TDG_S} + 2\|\delta^{1/2}\underline{\underline{\xi}}\|^2_{L^2(\mathcal{F}_h^{FS})}$$

$$|||(\omega_F, q, \omega_S, \underline{\underline{\xi}})|||^2_{TDG^*} := |||(\omega_F, q)|||^2_{TDG_F^*} + |||(\omega_S, \underline{\underline{\xi}})|||^2_{TDG_S^*} + \frac{1}{2}\|\delta^{-1/2}\underline{\underline{\xi}}\|^2_{L^2(\mathcal{F}_h^{FS})}$$

# ELASTO-ACOUSTIC SYSTEM

## COERCIVITY OF TREFFTZ-DG FORMULATION

$$\mathcal{A}_{TDG}((\omega_F, q, \omega_S, \underline{\underline{\xi}}); (\omega_F, q, \omega_S, \underline{\underline{\xi}})) = |||(\omega_F, q, \omega_S, \underline{\underline{\xi}})|||^2_{TDG}$$

# ELASTO-ACOUSTIC SYSTEM

## COERCIVITY OF TREFFTZ-DG FORMULATION

$$\mathcal{A}_{TDG}((\omega_F, q, \omega_S, \underline{\underline{\xi}}); (\omega_F, q, \omega_S, \underline{\underline{\xi}})) = |||(\omega_F, q, \omega_S, \underline{\underline{\xi}})|||^2_{TDG}$$

## CONTINUITY OF TREFFTZ-DG FORMULATION

$$|\mathcal{A}_{TDG}((v_F, p, v_S, \underline{\underline{\sigma}});(\omega_F, q, \omega_S, \underline{\underline{\xi}}))| \leq 2\,|||(v_F, p, v_S, \underline{\underline{\sigma}})|||_{TDG^*}\,|||(\omega_F, q\omega_S, \underline{\underline{\xi}})|||_{TDG}$$

$$|\ell_{TDG}(\omega_F, q, \omega_S, \underline{\underline{\xi}})| \leq \sqrt{2}\left[\|\rho_F^{1/2} v_{F0}\|^2_{L^2(\mathcal{F}_h^{0F})} + \|(\frac{1}{c_F^2\rho_F})^{1/2} p_0\|^2_{L^2(\mathcal{F}_h^{0F})}\right.$$
$$\left. + \|\rho_S^{1/2} v_{S0}\|^2_{L^2(\mathcal{F}_h^{0S})} + \|\underline{\underline{A}}^{1/2}\underline{\underline{\sigma}}_0\|^2_{L^2(\mathcal{F}_h^{0S})}\right]^{1/2}$$

# ELASTO-ACOUSTIC SYSTEM

## WELL-POSEDNESS

$$|||(v_F - v_{F\,h}, p - p_h, v_S - v_{S\,h}, \underline{\underline{\sigma}} - \underline{\underline{\sigma}}_h)|||_{TDG}$$

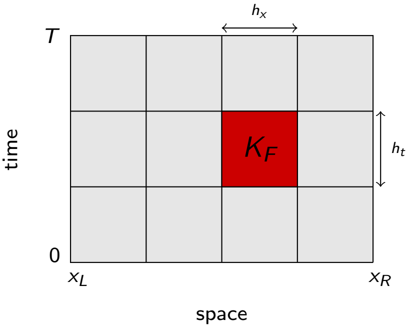
$$\leq 3 \inf_{(\omega_F, q, \omega_S, \underline{\underline{\xi}}) \in \mathbf{V}(\mathcal{T}_h)} |||(v_F - \omega_F, p - q, v_S - \omega_S, \underline{\underline{\sigma}} - \underline{\underline{\xi}})|||_{TDG^*}.$$

# IMPLEMENTATION OF THE ALGORITHM

## 1D ACOUSTIC SYSTEM EXAMPLE

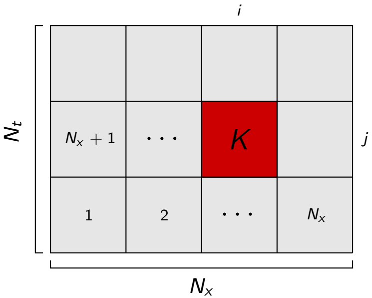
# IMPLEMENTATION OF THE ALGORITHM

## MESH AND ELEMENT NUMBERING



# IMPLEMENTATION OF THE ALGORITHM

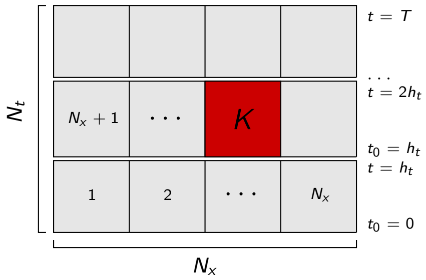
## MESH AND ELEMENT NUMBERING




$$K = (j - 1) \times N_x + i$$

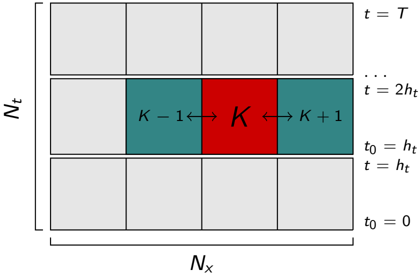
# IMPLEMENTATION OF THE ALGORITHM

## MESH AND ELEMENT NUMBERING



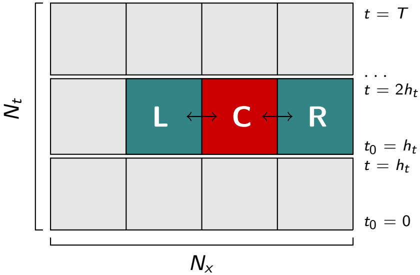
# IMPLEMENTATION OF THE ALGORITHM

## MESH AND ELEMENT NUMBERING



# IMPLEMENTATION OF THE ALGORITHM

## MESH AND ELEMENT NUMBERING



# IMPLEMENTATION OF THE ALGORITHM

## GLOBAL MATRIX ($1^{ST}$ LAYER)

$$
\begin{bmatrix}
C & R & & \cdots & & L \\
L & C & R & & & \\
& L & C & R & & \vdots \\
\vdots & & & \ddots & & \\
& & & L & C & R \\
R & & \cdots & & L & C
\end{bmatrix}
\times
\begin{bmatrix}
U_1 \\ U_2 \\ U_3 \\ \vdots \\ U_{N_x-1} \\ U_{N_x}
\end{bmatrix}
=
\begin{bmatrix}
U^0_1 \\ U^0_2 \\ U^0_3 \\ \vdots \\ U^0_{N_x-1} \\ U^0_{N_x}
\end{bmatrix}
$$

# IMPLEMENTATION OF THE ALGORITHM

## GLOBAL MATRIX ($1^{ST}$ LAYER)

$$\begin{bmatrix} \mathbf{M} \end{bmatrix} \times \begin{bmatrix} \mathbf{U} \end{bmatrix} = \begin{bmatrix} \mathbf{U}^0 \end{bmatrix}$$

# IMPLEMENTATION OF THE ALGORITHM

1. INITIATION

COMPUTE

$$\left[\mathbf{M}\right];\left[\mathbf{M}^{-1}\right];\left[\mathbf{U}^{0}\right]$$

2. PROPAGATION

FOR $t = 1 : N_t$

$$\left[\mathbf{U}\right] = \left[\mathbf{M}^{-1}\right] \times \left[\mathbf{U}^{0}\right]$$

$$\left[\mathbf{U}^{0}\right] = \left[\begin{array}{c}\text{projection}\\\text{operator}\end{array}\right] \times \left[\mathbf{U}\right]$$

ENDFOR

# IMPLEMENTATION OF THE ALGORITHM

## POLYNOMIAL BASIS (ORDER 3)

$$
\begin{aligned}
\phi_1^v &= 0 & \phi_1^p &= -c_F \\
\phi_2^v &= 1 & \phi_2^p &= 0 \\
\phi_3^v &= x & \phi_3^p &= -c_F^2 t \\
\phi_4^v &= c_F t & \phi_4^p &= -c_F x \\
\phi_5^v &= -\frac{x^2}{2} - \frac{c_F^2 t^2}{2} & \phi_5^p &= c_F^2 x t \\
\phi_6^v &= -c_F x t & \phi_6^p &= c_F \left(\frac{x^2}{2} + \frac{c_F^2 t^2}{2}\right) \\
\phi_7^v &= -\frac{x^3}{6} - \frac{x c_F^2 t^2}{2} & \phi_7^p &= c_F \left(\frac{c_F^3 t^3}{6} + \frac{x^2 c_F t}{2}\right) \\
\phi_8^v &= -\frac{c_F^3 t^3}{6} - \frac{x^2 c_F t}{2} & \phi_8^p &= c_F \left(\frac{x^3}{6} + \frac{x c_F^2 t^2}{2}\right)
\end{aligned}
$$

# IMPLEMENTATION OF THE ALGORITHM

## NUMERICAL TESTS

FIGURE 1: 1D Acoustic system.
Exact and numerical velocities $v_F(x, t = 1)$.
Periodical boundaries. $\alpha = \beta = 0$

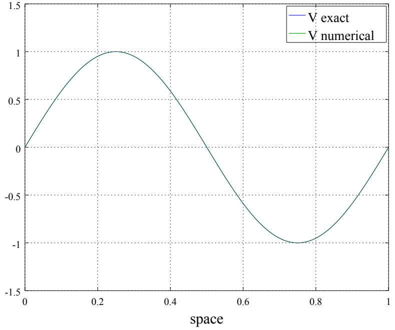
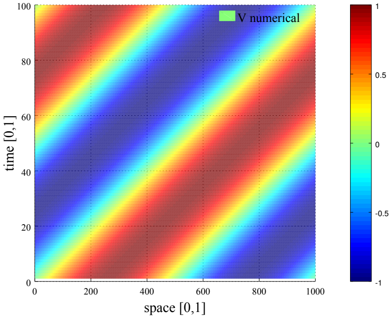



FIGURE 2: 1D Acoustic system.
Numerical velocity $v_F(x, t)$.
Periodical boundaries. $\alpha = \beta = 0$



# IMPLEMENTATION OF THE ALGORITHM

## NUMERICAL TESTS

FIGURE 3: 1D Acoustic system.
Exact and numerical velocities $v_F(x, t = 1)$.
Periodical boundaries. $\alpha = \beta = 0$

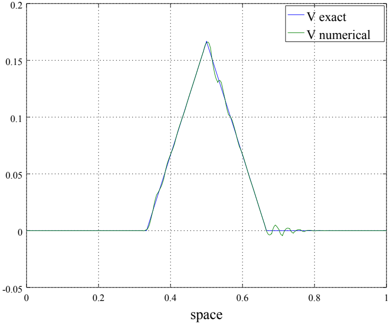
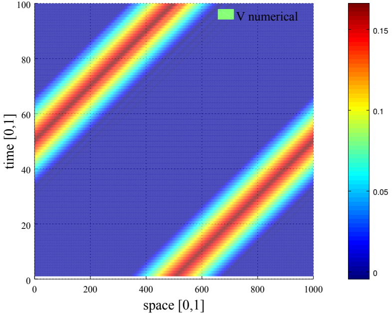



FIGURE 4: 1D Acoustic system.
Numerical velocity $v_F(x, t)$.
Periodical boundaries. $\alpha = \beta = 0$



# IMPLEMENTATION OF THE ALGORITHM

## NUMERICAL TESTS

FIGURE 5: 1D Acoustic system.
Exact and numerical velocities $v_F(x, t = 1)$.
Periodical boundaries. $\alpha = \beta = 0.5$

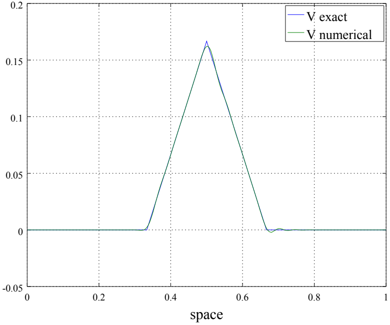
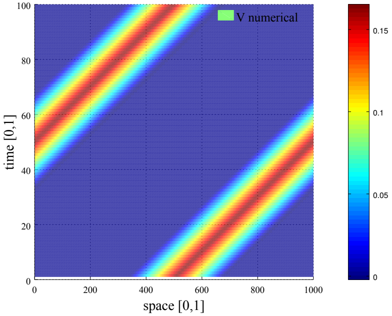



FIGURE 6: 1D Acoustic system.
Numerical velocity $v_F(x, t)$.
Periodical boundaries. $\alpha = \beta = 0.5$



# IMPLEMENTATION OF THE ALGORITHM

## NUMERICAL TESTS

FIGURE 7: 1D Acoustic system.
Convergence of velocity $v_F$ in function of cell size $h$.
Periodical boundaries. $\alpha = \beta = 0.0$

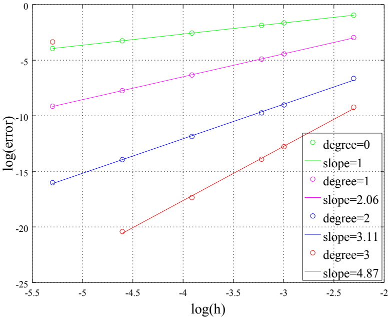
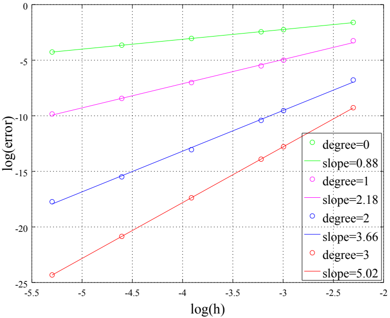



FIGURE 8: 1D Acoustic system.
Convergence of velocity $v_F$ in function of cell size $h$.
Periodical boundaries. $\alpha = \beta = 0.5$



# IMPLEMENTATION OF THE ALGORITHM

## NUMERICAL TESTS

FIGURE 9: 1D Acoustic system.
Numerical error in function of ratio $h_t/h_x$.
Periodical boundaries. $\alpha = \beta = 0.5$

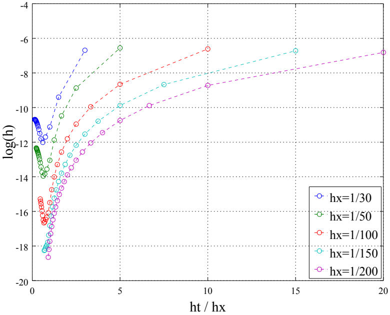
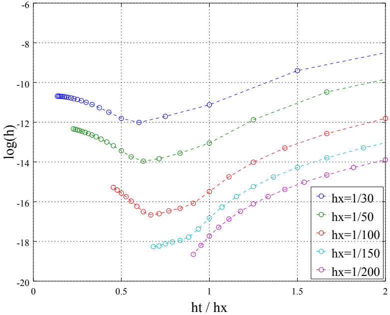



FIGURE 10: 1D Acoustic system.
Numerical error in function of ratio $h_t/h_x$.
Error minimum corresponds to $h_t/h_x = 1/(1+\alpha)$



# NUMERICAL RESULTS

## 2D SIMULATION

# NUMERICAL RESULTS

## 2D ACOUSTIC AND ELASTODYNAMIC SYSTEMS

FIGURE 11: 2D Acoustic system.
Convergence of velocity $v_F$ in function of cell size $h$

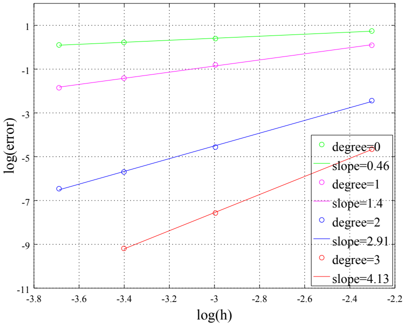
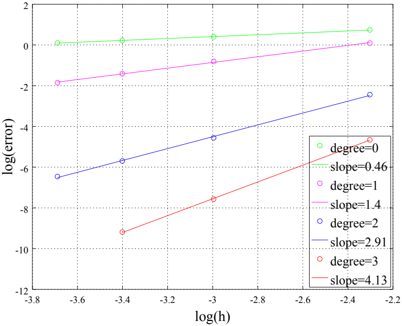


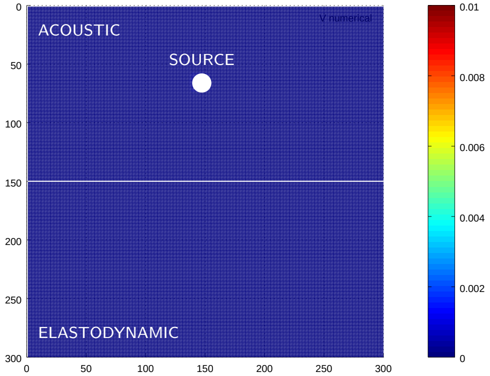

FIGURE 12: 2D Elastodynamic system.
Convergence of velocity $v_S$ in function of cell size $h$



# NUMERICAL RESULTS

## 2D ELASTO-ACOUSTIC SYSTEM

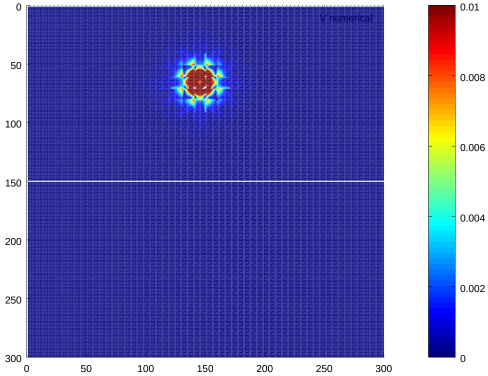
FIGURE 13: 2D Elasto-acoustic system.
Numerical velocity $v(x, 0.0)$. Dirichlet boundaries



# NUMERICAL RESULTS

## 2D ELASTO-ACOUSTIC SYSTEM

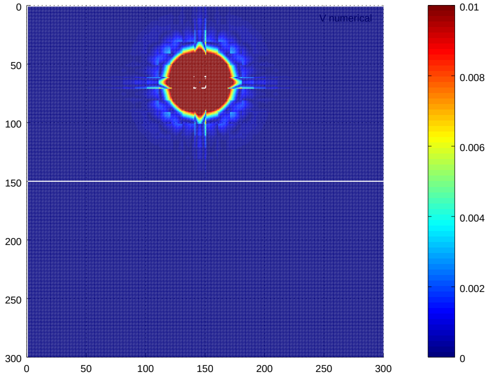
FIGURE 13: 2D Elasto-acoustic system.
Numerical velocity $v(x, 0.233)$. Dirichlet boundaries



# NUMERICAL RESULTS

## 2D ELASTO-ACOUSTIC SYSTEM

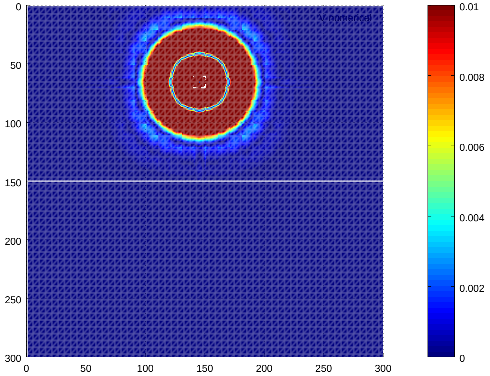
FIGURE 13: 2D Elasto-acoustic system.
Numerical velocity $v(x, 0.300)$. Dirichlet boundaries



# NUMERICAL RESULTS

## 2D ELASTO-ACOUSTIC SYSTEM

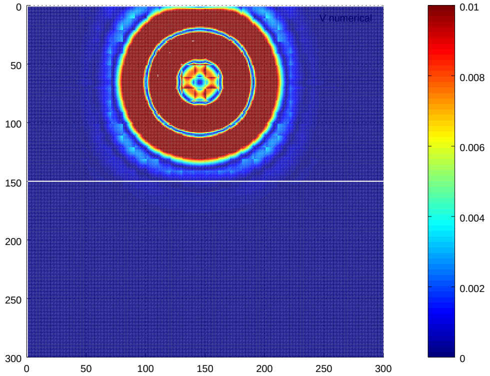
FIGURE 13: 2D Elasto-acoustic system.
Numerical velocity $v(x, 0.367)$. Dirichlet boundaries



# NUMERICAL RESULTS

## 2D ELASTO-ACOUSTIC SYSTEM

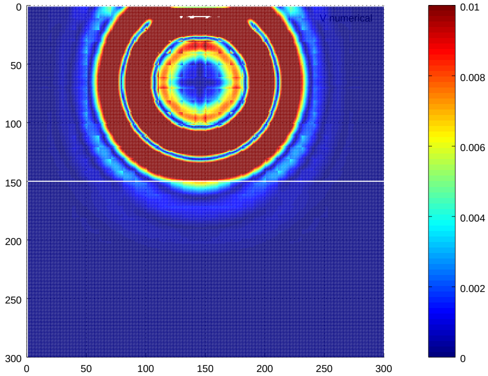
FIGURE 13: 2D Elasto-acoustic system.
Numerical velocity $v(x, 0.433)$. Dirichlet boundaries



# NUMERICAL RESULTS

## 2D ELASTO-ACOUSTIC SYSTEM

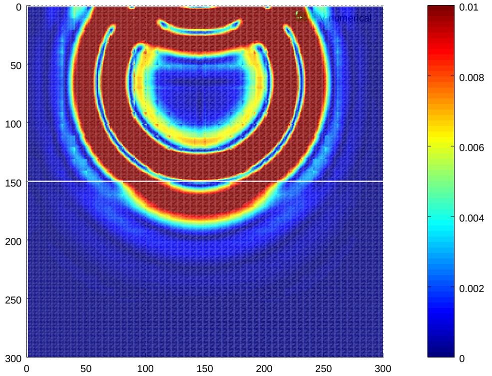
FIGURE 13: 2D Elasto-acoustic system.
Numerical velocity $v(x, 0.500)$. Dirichlet boundaries



# NUMERICAL RESULTS

## 2D ELASTO-ACOUSTIC SYSTEM

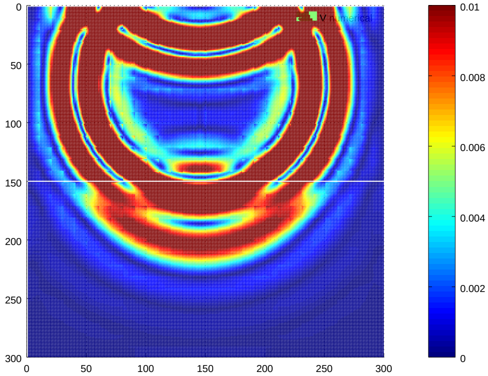
FIGURE 13: 2D Elasto-acoustic system.
Numerical velocity $v(x, 0.567)$. Dirichlet boundaries



# NUMERICAL RESULTS

## 2D ELASTO-ACOUSTIC SYSTEM

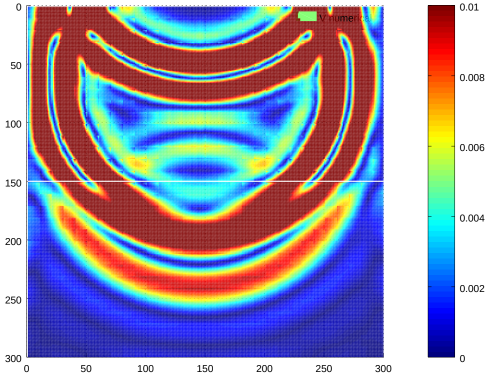
FIGURE 13: 2D Elasto-acoustic system.
Numerical velocity $v(x, 0.633)$. Dirichlet boundaries



# NUMERICAL RESULTS

## 2D ELASTO-ACOUSTIC SYSTEM

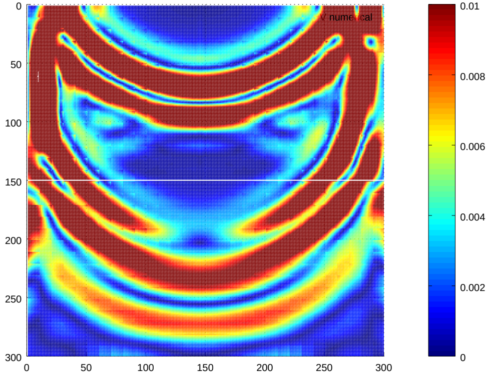
FIGURE 13: 2D Elasto-acoustic system.
Numerical velocity $v(x, 0.700)$. Dirichlet boundaries



# NUMERICAL RESULTS

## 2D ELASTO-ACOUSTIC SYSTEM

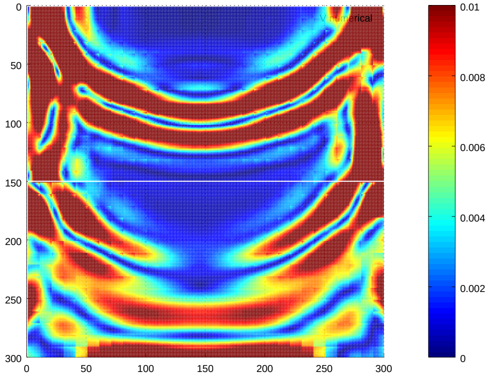
FIGURE 13: 2D Elasto-acoustic system.
Numerical velocity $v(x, 0.767)$. Dirichlet boundaries



# NUMERICAL RESULTS

## 2D ELASTO-ACOUSTIC SYSTEM

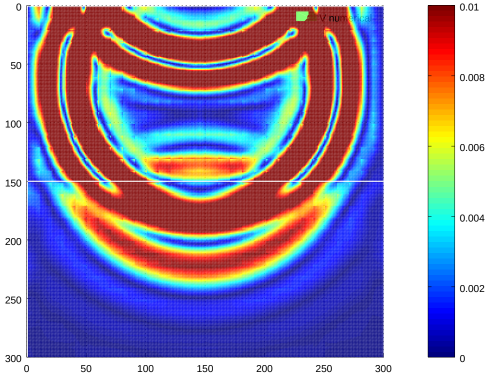
FIGURE 13: 2D Elasto-acoustic system.
Numerical velocity $v(x, 0.833)$. Dirichlet boundaries



# NUMERICAL RESULTS

## 2D ELASTO-ACOUSTIC SYSTEM

FIGURE 13: 2D Elasto-acoustic system.
Numerical velocity $v(x, 0.667)$. Dirichlet boundaries



# NUMERICAL RESULTS

## 2D ELASTO-ACOUSTIC SYSTEM

FIGURE 14: 2D Elasto-acoustic system. Numerical and analytical* seismograms for $v_x$. Source (150,75), receiver (195, 210)

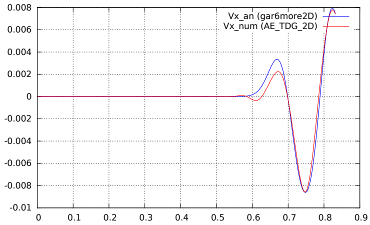
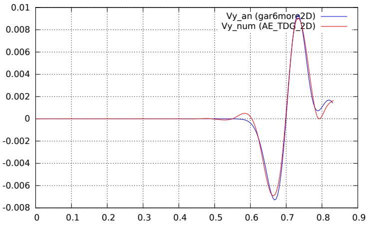



FIGURE 15: 2D Elasto-acoustic system. Numerical and analytical* seismograms for $v_y$. Source (150,75), receiver (195, 210)




* J.Diaz. *Gar6more2D.*

# CONCLUSION

# CONCLUSION

Implementation of the method for 2D acoustic, elastic, elasto-acoustic systems

Numerical tests, validation of the results by analytical solutions

Analytical and numerical study of efficiency of the method

Change-over between time-layers study (initial conditions)

# CONCLUSION

## ON-GOING WORK AND PERSPECTIVES

Change-over between time-layers study (source)

Alternative for the global matrix inversion

Boundary conditions

Hybrid Trefftz-DG + FVM coupling for the elasto-acoustics

# REFERENCES

J.S.Hesthaven T.Warburton. *Nodal discontinuous galerkin methods. Algorithms, analysis, and applications*. 2007.

F.Kretzschmar A.Moiola I.Perugia. *A priori error analysis of space-time trefftz discontinuous galerkin methods for wave problems.* 2015.

O.C.Zienkiewicz. *Trefftz type approximation and the generalized finite element method - history and developement.* 1997.

I.Herrera. *Trefftz method: a general theory.* 2000

A.Maciag. *Wave polynomials in elasticity problems.* 2006

# THANK YOU FOR ATTENTION!